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Exponentially Derived Switching Schemes for Inviscid Flow*

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A class of "exponential schemes" used for singular perturbation problems is taken and finite difference schemes for inviscid flow with shocks is derived from them. In particular, exponential schemes are formulated for steady viscous flow in a variable area duct using both a one-equation model (with the physical viscous terms) and the Euler equations (with artificial viscosity). Upon taking the limit as the viscosity coefficient goes to zero, "exponentially derived switching (EDS) schemes" are obtained which switch the direction of finite differencing based upon characteristic directions of the reduced problem. For the Euler equations some of the EDS schemes can be identified as flux vector splitting, the split coefficient matrix method, and a scheme of Huang. Some aspects of uniqueness of finite difference solutions are discussed.

1. INTRODUCTION

In this paper we study a certain class of finite difference schemes applied to steady state one-dimensional inviscid flow with shocks. We study schemes for duct flow using both a one-equation model and the Euler equations. The schemes we consider are derived from the so-called "exponential schemes" used for singular perturbation problems. Briefly, exponential schemes are constructed by piecing together locally exact exponential solutions of a modification of the original differential equation. For example, the modification may consist of the replacement of the coefficients of the original differential equation by piecewise constant approximations on subintervals of a mesh. These exponential schemes often yield better numerical approximations for thin boundary and interior layers than do polynomial schemes, and may also be free of "cell Reynolds number" restrictions. This basically means that solutions without numerical oscillations can be obtained when the mesh size is relatively large and the singular perturbation parameter ε is small.

To our knowledge, the first such exponential scheme was actually given in 1955 by Allen–Southwell, who applied it to compute two-dimensional incompressible viscous flow over a cylinder using a certain splitting technique [1]. Dennis [2] extended the

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Allen–Southwell technique to treat a slightly wider class of problems, and showed its second-order accuracy. In [3], Allen further discussed the undesirability of polynomial schemes for certain types of differential equations. Dennis [4] observed that the Allen–Southwell technique is free of a cell Reynolds number restriction, and that it reduces to simple upwinding in the limit of infinite Reynolds number. Similar ideas were discussed by Spalding [5].

As opposed to polynomial schemes, exponential schemes can yield uniform error estimates. This means that the discretization error goes to zero as some power of h, the mesh size, independent of the size of ε. In particular, the Allen–Southwell scheme has been analyzed [6–8] for a class of linear problems and shown to be uniformly first-order accurate. Recently, several investigators have applied exponential schemes to the driven cavity problem [9, 10]. However, it is still not clear whether such schemes can be effectively used for a spectrum of multidimensional fluid problems.

Typically, for fluid dynamics applications, the small parameter ε is the viscosity, and one is interested in both the small ε case (yielding boundary or interior layers) and the $\varepsilon \to 0$ limit (inviscid flow with shocks). In the present paper, we investigate what happens when the formal $\varepsilon \to 0$ limit of a class of exponential schemes is applied to the reduced problem of shocked inviscid flow. We call such schemes "exponentially derived switching schemes" (EDS schemes) since they are the limit of exponentially based schemes and they involve switching the direction of finite differencing ("upwinding" in some sense) based on the computed solution itself [26, p. 183]. We note that Engquist and Osher [11–14] have introduced and analyzed a different class of switching schemes.

We first motivate the derivation of EDS schemes by considering the steady state Burgers equation. We then consider a one-equation model for inviscid flow in an expanding duct with a standing shock. For this equation we obtain numerical results for two different EDS schemes. For one of these schemes an existence and uniqueness result has been established [15].

Considerations of uniqueness are of interest to use for the following reasons. It is known that for many viscous fluid dynamics problems the partial differential equations possess a unique solution, but it is known that in the inviscid limit there may often be many weak solutions (e.g., solutions containing discontinuities). In the latter case an additional restriction, called the entropy condition, must be enforced in order to obtain the limit of viscous solutions as $\varepsilon \to 0$. The uniqueness situation is even more complicated for the approximating finite difference problem. Here, consistent approximations to either the viscous or inviscid problem may possess more than one finite difference solution [16]. In some cases it is difficult to identify which solution best approximates the flow field. It is therefore advantageous to compute with a scheme that possesses only one solution, or at least only one solution with the correct properties. We emphasize, however, that taking the limit of exponential schemes for the viscous problem certainly does not guarantee that the resulting inviscid scheme will provide a unique solution.

Finally, we consider EDS schemes for the system of Euler equations describing steady one-dimensional duct flow. Again we obtain numerical results for several EDS schemes and show that the flux vector splitting scheme [17], the split coefficient method [18] and a recent scheme of Huang [19] are actually EDS schemes. For time-dependent problems this indicates that the switching properties of EDS schemes are related to characteristic directions.

2. BURGERS EQUATION

In order to motivate the exponentially derived switching (EDS) schemes for one-dimensional duct flow we first consider the steady state Burgers equation

$$\begin{gathered} u u_x - \varepsilon u_{xx} = 0 \\ u(0) = 1, \qquad u(1) = -1 \end{gathered} \tag{2.1}$$

whose exact solution is given by

$$u(x) = -K \tanh(K(x - .5)/(2\varepsilon))$$

where $K \tanh(K/4\varepsilon) = 1$.

The corresponding inviscid ($\varepsilon = 0$) equation is

$$u u_x = (u^2/2)_x = 0 \tag{2.2}$$

with the same boundary conditions as (2.1). Equation (2.2) does not admit smooth solutions satisfying the boundary conditions, but does admit generalized (weak) solutions with jump discontinuities. In fact, (2.2) does admit a solution

$$u(x) = \begin{cases} 1 & x < \frac{1}{2} \\ -1 & x > \frac{1}{2} \end{cases} \tag{2.3}$$

which is the unique inviscid limit of solutions of (2.1).

We now define two formally second-order accurate exponential schemes for (2.1) and use them to derive EDS schemes for (2.2). They are

$$\frac{u_i}{2h}(u_{i+1} - u_{i-1}) - \frac{\varepsilon \Gamma_i}{h^2}(u_{i+1} - 2u_i + u_{i-1}) = 0 \tag{2.4}$$

and

$$\frac{1}{4h}(u_{i+1}^2 - u_{i-1}^2) - \frac{\varepsilon}{h^2}(\Gamma_{i+1} u_{i+1} - 2\Gamma_i u_i + \Gamma_{i-1} u_{i-1}) = 0 \tag{2.5}$$

where $\Gamma_i = (u_i h/2\varepsilon)$ coth $(u_i h/2\varepsilon)$. Scheme (2.4) is an adaptation of a scheme of Allen and Southwell [1], and (2.5) is a modification of (2.4) in conservation form [20].

We proceed to construct EDS schemes for (2.2) by taking the formal limit of (2.4) and (2.5) as $\varepsilon \to 0$. Noting that $\lim_{\varepsilon \to 0} y \coth(y/\varepsilon) = y \operatorname{sgn}(y) = |y|$, we obtain respectively

$$u_i(u_{i+1} - u_{i-1}) - |u_i| (u_{i+1} - 2u_i + u_{i-1}) = 0 \tag{2.6}$$

and

$$u_{i+1}^2 - u_{i-1}^2 - 2(|u_{i+1}| u_{i+1} - 2|u_i| u_i + |u_{i-1}| u_{i-1}) = 0. \tag{2.7}$$

We observe that the viscous terms do not vanish in the $\varepsilon \to 0$ limit and therefore represent a kind of artificial viscosity; indeed, Eqs. (2.6) and (2.7) may be regarded as difference schemes for $uu_x = (h/2)|u| u_{xx}$ and $uu_x = (h/2)(|u| u)_{xx}$, respectively. We say that (2.6) has "one switching point" since it depends on the sign of u at the single point i. Similarly (2.7) has "three switching points." We note that (2.6) reduces to

$$\begin{aligned} u_i(u_i - u_{i-1}) &= 0 \qquad \text{if} \quad u_i > 0 \\ u_i(u_{i+1} - u_i) &= 0 \qquad \text{if} \quad u_i < 0 \end{aligned} \tag{2.8}$$

which is a standard "upwind" scheme with the differencing direction switching on the sign of u_i. On the other hand, (2.7) is not an upwind scheme. While (2.6) possesses sharp shock profiles, it also has multiple solutions. In particular, among other solutions it admits the family of solutions $u_0 = \cdots = u_{i_0} = 1$, $u_{i_0+1} = \cdots = u_{n+1} = -1$ for any $0 \leqslant i_0 \leqslant n$. By contrast, it can be shown that scheme (2.7) possesses a unique solution whose components are monotonically decreasing and antisymmetric with respect to $x = \frac{1}{2}$. We note that the existence, uniqueness, and properties of scheme (2.5) for the related viscous problem are established in [20] using the theory of M-functions. So in a sense we may think of the uniqueness of the solution of (2.7) as being "inherited" from scheme (2.5). We note that certain schemes for time-dependent problems, namely, monotone schemes in conservation form, have been shown by Crandal and Majda [21] to possess a solution converging to the unique inviscid solution satisfying the entropy condition. We note that the time-dependent version of (2.7)

$$u_i^{k+1} = u_i^k - \frac{\Delta t}{4h}(u_{i+1}^{k2} - u_{i-1}^{k2}) + \frac{\Delta t}{2h}(|u_{i+1}^k| u_{i+1}^k - 2|u_i^k| u_i^k + |u_{i-1}^k| u_{i-1}^k) \tag{2.9}$$

is a weakly monotone scheme with restricted CFL condition $|u_i| \Delta t/h < \frac{1}{2}$.

Numerical results for (2.6) and (2.7) with $n = 9, 19, 39$ ($h = \frac{1}{10}, \frac{1}{20}, \frac{1}{40}$) are given in Figs. 2–1 and 2–2, respectively. As scheme (2.6) possesses multiple solutions, we have shown only the antisymmetric solution most like those for scheme (2.7). We note that, although (2.7) has a unique solution, it tends to give somewhat smeared

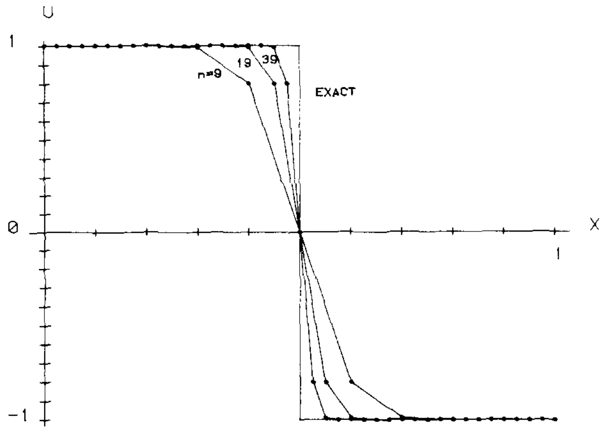

FIG. 2–1. Comparison of solutions to scheme (2.6) with exact solution. Burgers equation. One switching point scheme.

profiles (approximately a "five point shock"). Also, if we write (2.7) as a system of difference equations $F_i(u) = 0$, $i = 1, \ldots, n$, then $\sum_{i=1}^{n} F_i^2$ appears to have many local minima. Thus when a time asymptotic method is used to solve (2.7), often the iterates will appear to be converging to a local minimum, and then after many iterations will eventually move away from the minimum and converge to the unique solution. Similarly, any particular solution to scheme (2.6) is not easy to get because of the presence of multiple solutions.

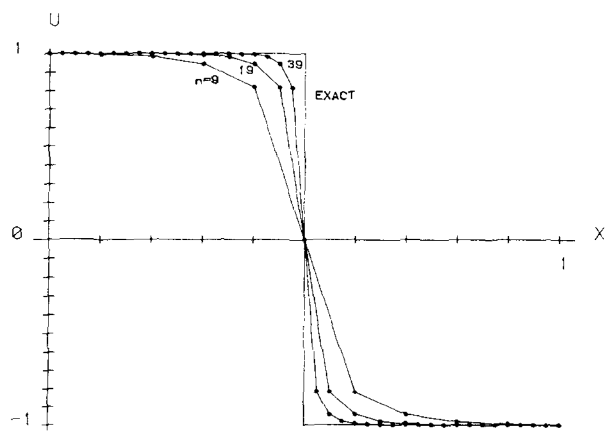

FIG. 2–2. Comparison of solutions to scheme (2.7) with exact solution. Burgers equation. Three switching point scheme.

3. Duct Flow—One-Equation Model

A. *Formulation*

We now consider one-dimensional flow in a duct of variable cross-sectional area $A(x)$, $0 \leqslant x \leqslant x_{\max}$. First formulating the problem in the usual system form for steady, non-heat-conducting, viscous flow, we have

$$\textit{Viscous system:} \qquad \mathscr{F}_x + \mathscr{G} = \varepsilon \mathscr{V}_x \tag{3.1}$$

where

$$\mathscr{F} = A \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (\rho E + p) u \end{bmatrix}, \qquad \mathscr{G} = \begin{bmatrix} 0 \\ -pA_x \\ 0 \end{bmatrix}, \qquad \mathscr{V} = Au_x \begin{bmatrix} 0 \\ 1 \\ u \end{bmatrix}$$

and ρ is density, u is velocity, $E = e + u^2/2$, where e is specific internal energy, p is pressure, and ε is the viscosity coefficient (here assumed constant). The pressure is given by the perfect gas equation of state, $p = (\gamma - 1)\rho e$, where the constant γ is the ratio of specific heats. Boundary conditions are discussed later. As before, we consider the viscous problem (3.1) as a tool to get schemes for the inviscid problem, which is

$$\textit{Inviscid system:} \qquad \mathscr{F}_x + \mathscr{G} = 0. \tag{3.2}$$

It is possible to directly integrate the first and third components of (3.1) to give $\rho u A = c$, $e + u^2/2 + p/\rho - \varepsilon u_x/\rho = H$, where c and H are constants to be evaluated at a boundary (where we shall assume $u_x = 0$). These results can be combined with the second component of (3.1) to give a single equation for u as

$$\textit{Viscous model:} \qquad \tilde{q} u_x - \varepsilon u_{xx} + \tilde{g} = 0 \tag{3.3}$$

where $\tilde{q} = \gamma(\tilde{D} - \tilde{E}/u^2)/A$, $\tilde{D} = (\gamma + 1) c/(2\gamma) - \varepsilon A'$, $\tilde{E} = (\gamma - 1) cH/\gamma$, and $\tilde{g} = -(\gamma - 1) c(H/u - u/2) A'/A^2$. Taking the inviscid limit ($\varepsilon \to 0$) and rewriting in conservation form, we obtain

$$\textit{Inviscid model:} \qquad r_x + g = 0 \tag{3.4}$$

where $r = \gamma(Du + E/u)/A$, $D = (\gamma + 1) c/(2\gamma)$, $E = (\gamma - 1) cH/\gamma$, $g = c\gamma u A'/A^2$ and D, E and H are now constants. We consider the single equation models (3.3) and (3.4) in this section, returning to the system formulation (3.1) and (3.2) in Section 4. We note that in the inviscid case $\varepsilon = 0$ it can be shown by a simple computation that the velocity u is sonic (Mach one) at $u = a \equiv (E/D)^{1/2}$.

We restrict ourselves to the case where the flow is from left to right ($u > 0$) and the duct area $A(x)$ is increasing ($A' > 0$). To obtain a shocked flow we specify supersonic inflow ($u > a$) at $x = 0$ and subsonic outflow ($u < a$) at $x = x_{\max}$. A discussion of the boundary conditions appropriate for solving (3.2) in this case is given in [22].

For the single equation models (3.3) and (3.4), it appears to be correct to specify $u(0)$ and $u(x_{max})$.

As an aside, we note that a useful model for testing time-dependent numerical methods can be formed by taking

$$u_t + r(u, x)_x + g = 0 \tag{3.4a}$$

which incorporates a non-physical time evolution, but reduces to the correct steady state physics. This model provides a considerably more demanding test of numerical schemes than does Burgers equation.

B. *Exponential Schemes for the Viscous Problem*

We formulate two types of EDS schemes for (3.4) by first writing approximations (analogous to (2.4) and (2.5) for Burgers equation) for the finite ε case, Eq. (3.3), and then taking the $\varepsilon \to 0$ limit. We first restate the "non-conservation" form (3.3)

$$\tilde{q}u_x - \varepsilon u_{xx} + \tilde{g} = 0 \tag{3.5}$$

and then rewrite in "conservation form" (making an approximation by neglecting the dependence of $\tilde{D}$ on x; this dependence disappears in the $\varepsilon \to 0$ limit)

$$\tilde{r}_x - \varepsilon u_{xx} + G = 0 \tag{3.6}$$

where $\tilde{r} = \gamma(\tilde{D}u + \tilde{E}/u)/A$ and $G = \tilde{g} + \tilde{r}A'/A$.

As before, we discretize as $h = x_{max}/(n+1)$, $x_i = ih$, and consider $u(x_i) = u_i$ to be an approximation to u at x_i. We set $u_0 = u(x=0)$ and $u_{n+1} = u(x = x_{max})$. An Allen–Southwell type approximation for (3.5) (patterned after (2.4) for Burgers equation) is

$$\frac{\tilde{q}_i}{2h}(u_{i+1} - u_{i-1}) - \frac{\varepsilon}{h^2}\Gamma_i(u_{i+1} - 2u_i + u_{i-1}) + \tilde{g}_i = 0 \tag{3.7}$$

where

$$\Gamma_i = \frac{\tilde{q}_i h}{2\varepsilon}\coth\left(\frac{\tilde{q}_i h}{2\varepsilon}\right).$$

Alternatively, a scheme in conservation form for (3.6) analogous to (2.5) is given by

$$\frac{1}{2h}(\tilde{r}_{i+1} - \tilde{r}_{i-1}) - \frac{\varepsilon}{h^2}(\Gamma_{i+1}u_{i+1} - 2\Gamma_i u_i + \Gamma_{i-1}u_{i-1}) + G_i = 0. \tag{3.8}$$

A further variation of (3.8) is obtained by noting that $u_{xx} = (u-a)_{xx}$ for a = constant. Here we take a to be the previously mentioned value of u at which the inviscid flow is exactly sonic, $a = (E/D)^{1/2}$. This variation is necessary for the proof of existence and uniqueness given in [15] to go through. This alternate scheme for the viscous problem (which will lead to our principal EDS scheme) is

$$\frac{1}{2h}(\tilde{r}_{i+1}-\tilde{r}_{i-1})-\frac{\varepsilon}{h^2}(\Gamma_{i+1}(u_{i+i}-a)-2\Gamma_i(u_i-a)$$
$$+\Gamma_{i-1}(u_{i-1}-a))+G_i=0. \tag{3.9}$$

C. *EDS Schemes for the Inviscid Problem*

We now take the formal $\varepsilon \to 0$ limit of schemes (3.7) and (3.9) to obtain exponentially derived switching (EDS) schemes for the inviscid model (3.4). Noting first that

$$\lim_{\varepsilon\to 0}(\varepsilon\Gamma_i)=\frac{q_i h}{2}\operatorname{sgn}(q_i)=\frac{h}{2}|q_i|, \qquad q=\gamma(D-E/u^2)/A,$$

we get from (3.7) the EDS scheme

$$\frac{q_i}{2h}(u_{i+1}-u_{i-1})-\frac{1}{2h}|q_i|(u_{i+1}-2u_i+u_{i-1})$$
$$+\left(G-\frac{rA'}{A}\right)_i=0 \tag{3.10}$$

and from (3.9) the EDS scheme (with "shifting" by the sonic velocity a)

$$\frac{1}{2h}(r_{i+1}-r_{i-1})-\frac{1}{2h}(|q_{i+1}|(u_{i+1}-a)-2|q_i|(u_i-a)$$
$$+|q_{i-1}|(u_{i-1}-a))+g_i=0. \tag{3.11}$$

Scheme (3.10) has one switching point and scheme (3.11) has three switching points. We note that (3.11) is not an upstream scheme in the sense of Harten, Lax and van Leer [23]. However, (3.11) is conservative and also approximates

$$r_x-(h/2)(|q|(u-a))_{xx}+g=0.$$

D. *Numerical Results*

We now exhibit numerical results for duct flow obtained with the one switching point EDS scheme (3.10) and the three switching point EDS scheme (3.11). We use the model problem introduced in [22] in which the duct area is given by $A(x)=1.398+0.347\tanh(0.8x-4)$, $x_{\max}=10$. Although we specify only $u_0=u(0)$ and

$u_{n+1} = u(x_{max})$ in obtaining our results for the single-equation model, we give more information here for reference later.

$$\text{Duct flow problem:} \quad \begin{aligned} u(0) &= 1.299 \\ \rho(0) &= 0.502 \\ e(0) &= 1.897 \\ u(x_{max}) &= 0.505 \\ \rho(x_{max}) &= 0.776 \\ \gamma &= 1.4. \end{aligned} \tag{3.12}$$

For these conditions, an exact solution to (3.4), and hence to (3.2), can be constructed in which a single shock of shock Mach number 1.71 stands at $x = 4.816$. Our numerical results in this section were obtained by using Newton's method to solve the system of finite difference equations, (3.10) or (3.11), which hold for $i = 1, 2, \ldots, n$, subject to the specified boundary values u_0 and u_{n+1}. Our initial guess for Newton's method was a linear interpolation between u_0 and u_{n+1}.

Our results for $n = 9$, 19 and 39 obtained with schemes (3.10) and (3.11) are shown in Figs. 3–1 and 3–2, respectively. In each case, the exact solution of (3.2) is shown for comparison. In our opinion, the results given in Fig. 3–2 for the three switching point scheme (3.11) are too "smeared out" and do not represent particularly good approximations to the exact solution. However, this scheme enjoys the property that only one solution exists with all $u_i > 0$ [15]. (Although solutions do indeed exist with at least one $u_i < 0$, such "spurious" solutions are readily identifiable). Also there are no overshoots or undershoots (oscillations). By contrast, the

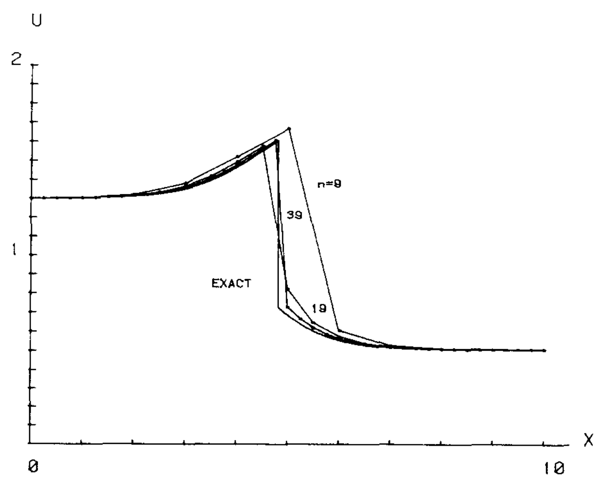

FIG. 3–1. Comparison of solutions to scheme (3.10) with exact solution. Duct flow—one-equation model. One switching point scheme.

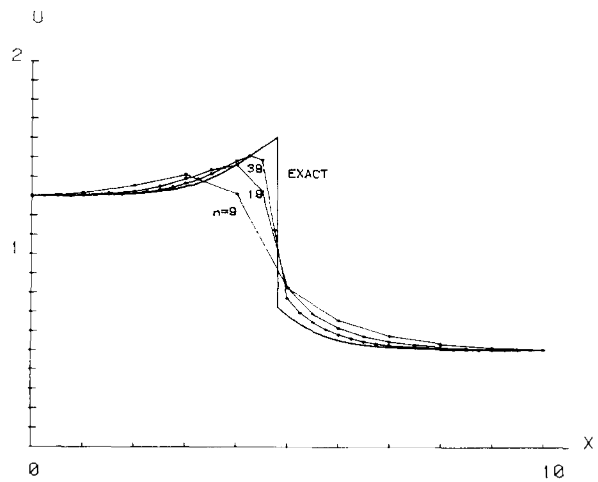

FIG. 3–2. Comparison of solutions to scheme (3.11) with exact solution. Duct flow—one-equation model. Three switching point scheme.

solutions shown in Fig. 3–1 for the one switching point scheme (3.10) are "sharper" than those for (3.11). However, more than one solution with $u_i > 0$ exists, for fixed n, a shown in Fig. 3–3. This type of multiple solutions is particularly troublesome, since (without knowledge of the exact solution) it is difficult to distinguish which solution is the best approximation. We note that, for comparison purposes, the popular MacCormack predictor-corrector scheme [24] (with a little artificial viscosity added [22]) when applied to (3.2) gives results whose sharpness is comparable to scheme (3.10) but with overshoots and undershoots.

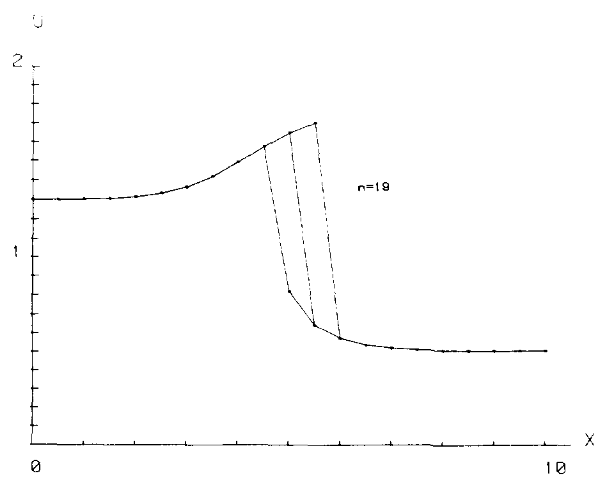

FIG. 3–3. Multiple solutions to scheme (3.10). Duct flow—one-equation model. One switching point scheme. Multiple solutions.

Finally, we have also done some computation for a stronger shock case, the results of which are qualitatively and quantitatively similar and are thus not presented. For this case we note that, using Newton's method, the unique solution to scheme (3.11) with all $u_i > 0$ is not easy to obtain. We succeeded only by using "damping" and starting with a "super" solution. This super solution was obtained using the combinatorial method of [25].

4. Duct Flow—System Formulation

We now return to consider three specific exponentially derived switching schemes for the system form of the gas dynamics equations, (3.2). While we consider these schemes from the point of view of steady flow, we will show that they are directly related (and sometimes identical) to some already known schemes for time-dependent gasdynamics.

A. *One Switching Point Scheme*

We first consider a variant of a scheme given by Doolan, Miller and Schilders (DMS), Ref. [26], for the system:

$$a(x)\, u_x + b(x)\, u - f(x) = \varepsilon I u_{xx} \tag{4.1}$$

where u and f are n-vectors and a and b are $n \times n$ matrices. The DMS scheme, which is for linear problems without turning points (changes in sign of the eigenvalues of a), is given (for the case $b(x) = 0$) by

$$\begin{aligned} \frac{a_i}{2h}(u_{i+1} - u_{i-1}) - f_i &= \frac{\varepsilon \Gamma_i}{h^2}(u_{i+1} - 2u_i + u_{i-1}) \\ \Gamma_i &= \frac{a_i h}{2\varepsilon} \coth\left(\frac{a_i h}{2\varepsilon}\right). \end{aligned} \tag{4.2}$$

This method is apparently a straightforward generalization of the Allen–Southwell scheme, Eq. (2.4), and has to our knowledge seen little application. Attempts to apply this scheme to the gasdynamics equations, (3.1), encounter several difficulties. Of course, (3.1) is nonlinear. Also, for the case in which a "shock" occurs, there is a turning point. Furthermore, when (3.1) is rewritten in the form (4.1), we obtain

$$\bar{A} U_x + \mathscr{G} = \varepsilon \bar{B} U_{xx} \tag{4.3}$$

where U is the "conservation vector" $U = A(\rho, \rho u, \rho E)^T$, $\bar{A} = [\partial \mathscr{F} / \partial U]$, and $B = [\partial \mathscr{G} / \partial U]$. Unfortunately, $\bar{B}$ is not invertible and thus (4.3) cannot be put in the form (4.1).

We rectify this situation by considering system (3.1) with the physical viscous term εU_x replaced by the artificial viscous term $\varepsilon I U_{xx}$. Smoller and Conley [27, 28] have investigated which artificial viscous terms will lead to the correct limit solutions as $\varepsilon \to 0$. Foy [29] has shown that, for weak shocks, the addition of the artificial viscous term $\varepsilon I U_{xx}$ to a hyperbolic system of conservation laws results in a system which gives the correct inviscid behavior in the limit $\varepsilon \to 0$. The system we will consider is thus

$$\bar{A} U_x + \mathscr{G} = \varepsilon I U_{xx} \tag{4.4}$$

where

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 \\ (\gamma-3)\,u^2/2 & (3-\gamma)\,u & \gamma-1 \\ u(-\gamma e + u^2(\gamma-2)/2) & \gamma e + u^2(3-2\gamma)/2 & \gamma u \end{bmatrix}.$$

We note that $\bar{A}$ is certainly diagonalizable, since when the term U_t is added to the left side of (4.3) and $\varepsilon = 0$, the resulting time-dependent gas dynamics system is hyperbolic. This diagonalization is crucial in evaluating the matrix hyperbolic cotangent given in (4.2), which is done as follows. Let $\bar{A} = PDP^{-1}$, where D is a diagonal matrix with the eigenvalues of $\bar{A}$ on the diagonal, $D = \mathrm{Diag}(u, u - a^*, u + a^*)$, where $a^* = (\gamma(\gamma-1)\,e)^{1/2}$ is the speed of sound. P is a matrix whose columns are the corresponding right eigenvectors of $\bar{A}$. Then $\coth \bar{A} = P(\coth D)\,P^{-1}$, where $\coth D = \mathrm{Diag}(\coth u, \coth(u - a^*), \coth(u + a^*))$.

Although scheme (4.2) was not designed for nonlinear problems with turning points, we have tried taking the $\varepsilon \to 0$ limit of this scheme and applying the resulting exponentially derived switching scheme

$$\begin{gathered} \frac{\bar{A}_i}{2h}(U_{i+1} - U_{i-1}) + \mathscr{G}_i = \frac{1}{2h} Q_i (U_{i+1} - 2U_i + U_{i-1}) \\ Q_i = (P\,|D|\,P^{-1})_i \end{gathered} \tag{4.5}$$

to the problem (3.12). Here $|D| = \mathrm{Diag}(|u_i|, |u_i - a^*|, |u_i + a^*|)$. We note that for supersonic flow $u > a^*$, (4.5) reduces to pure "upwinding" (backward differencing) since $|D| = D$ and thus $Q_i = (PDP^{-1})_i = \bar{A}_i$.

Scheme (4.5) is a one switching point scheme that switches the nature of the finite differencing based only upon the signs of the eigenvalues of $\bar{A}$ at the one point i. Indeed, these eigenvalues are just the characteristic speeds $dx/dt = u$, $u - a^*$, $u + a^*$ for the associated hyperbolic time-dependent problem. This scheme is in fact related to the split-coefficient-matrix (SCM) method [18] as follows. If (3.1) is modified to the time-dependent non-conservation form $W_t + D_0 W_x + E_0 = \varepsilon I W_{xx}$, where $W = (\rho, u, e)^T$ and $D_0 = [\partial \mathscr{F}/\partial Q]$, an Allen–Southwell type approximation is made, and the $\varepsilon \to 0$ limit is taken, the SCM method is recovered.

Not surprisingly, (4.5) does not appear to yield decent physical approximations to problem (3.12). This failure is apparently related to the fact that (4.5) does not conserve the correct physical quantities across a shock. When a time-dependent computation using (4.5) (plus the physically correct time-dependent term U_t) was started from the exact solution, it did not coverge to any reasonably approximation. We have also been unsuccessful in obtaining decent results for our problem with the aforementioned SCM method. (In these tests, the system boundary conditions used were those discussed in Section 3A.)

B. *Three Switching Point Scheme*

In view of the above results, we modify (4.5) in a manner similar to that which led from the approximation (2.4) to the approximation (2.5) for Burgers equation. Namely, our new scheme for (4.4) is given as

$$\frac{1}{2h}(\mathscr{F}_{i+1}-\mathscr{F}_{i-1})+\mathscr{G}_i=\frac{\varepsilon}{h^2}(\Gamma_{i+1}U_{i+1}-2\Gamma_iU_i+\Gamma_{i-1}U_{i-1})$$
$$\Gamma_i=\frac{\bar{A}_ih}{2\varepsilon}\coth\left(\frac{\bar{A}_ih}{2\varepsilon}\right) \tag{4.6}$$

and in the limit $\varepsilon\to 0$ we get the corresponding EDS scheme

$$\frac{1}{2h}(\mathscr{F}_{i+1}-\mathscr{F}_{i-1})+\mathscr{G}_i=\frac{1}{2h}[(QU)_{i+1}-2(QU)_i+(QU)_{i-1}]. \tag{4.7}$$

Scheme (4.7) has three switching points since it depends on the eigenvalues of $\bar{A}$ at the three points $i-1$, i, $i+1$. Using the same uniform spatial discretization described before, the system boundary conditions previously discussed, and a time-dependent computation, we obtained the steady state solutions shown in Fig. 4–1 for $n=9$, 19 and 39.

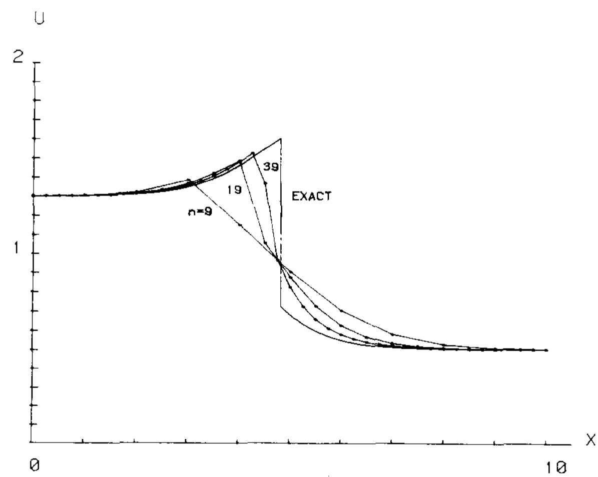

FIG. 4–1. Comparison of solutions to scheme (4.7) with exact solution. Duct flow—system. Three switching point scheme.

Again, we feel that the computed solutions are quite smeared out and do not represent particularly good approximations. However, the computed solution has no overshoots. We note in passing that if $\mathscr{G}_i$ in (4.7) is replaced by $\frac{1}{2}(\mathscr{G}_{i+1}+\mathscr{G}_{i-1})$, slightly better results can be obtained.

It is quite interesting to note that the three switching point scheme given in (4.7) is exactly equivalent to the flux vector splitting (FVS) method of Steger and Warming for one particular splitting of the eigenvalues and one particular spatial discretization [17, p. 270]. To see this we must use the fact, crucial to FVS, that the gas dynamics equations are homogeneous of degree one. Among other things, this implies that $\mathscr{F}_U U = \bar{A}U = \mathscr{F}$. Using this homogeneity property we may rewrite (4.7) as

$$\frac{1}{2h}\{(\mathscr{F}-\mathscr{S})_{i+1}-(\mathscr{F}-\mathscr{S})_i+(\mathscr{F}+\mathscr{S})_i-(\mathscr{F}+\mathscr{S})_{i-1}\}+\mathscr{G}_i=0 \quad (4.8)$$

where $S = QU$. We can then identify this as the FVS scheme

$$\frac{1}{h}(\Delta\mathscr{F}^- + \nabla\mathscr{F}^+)+\mathscr{G}_i=0$$

where

$$\begin{aligned}\mathscr{F}^{\pm} &= \tfrac{1}{2}(\mathscr{F}\pm\mathscr{S})=\tfrac{1}{2}(PDP^{-1}U \pm P|D|P^{-1}U)\\ &= P\left(\frac{D\pm|D|}{2}\right)P^{-1}U\end{aligned}$$

and Δ and ∇ are forward and backward difference operators.

C. *Two Switching Point Schemes*

Finally, we consider two alternatives to (4.2) as approximation schemes for (4.1). The first is a scheme due to El-Mistikawy and Werle [30] which has been analyzed in [31]. When extended to approximate system (4.1), this scheme (with $b(x)=0$) is

$$\begin{aligned}&\frac{1}{2}\left\{a_{i+1/2}\frac{(u_{i+1}-u_i)}{h}+a_{i-1/2}\frac{(u_i-u_{i-1})}{h}\right\}-\frac{1}{2}(f_{i+1}+f_{i-1})\\ &=\frac{\varepsilon}{h^2}\{\gamma_{i+1/2}(u_{i+1}-u_i)-\gamma_{i-1/2}(u_i-u_{i-1})\} \qquad (4.9)\end{aligned}$$

where

$$a_{i\pm1/2}=(a_i+a_{i\pm1})/2 \qquad \text{and} \qquad \gamma_{i\pm1/2}=\frac{a_{i\pm1/2h}}{2\varepsilon}\coth\left(\frac{a_{i\pm1/2h}}{2\varepsilon}\right).$$

When this scheme is applied to gas dynamics system (4.4) and the $\varepsilon \to 0$ limit is taken, we obtain the two switching point EDS scheme

$$\frac{1}{2h}\left[(\bar{A}_{i+1/2}-Q_{i+1/2})(U_{i+1}-U_i)+(\bar{A}_{i-1/2}+Q_{i-1/2})(U_i-U_{i-1})\right] + \frac{1}{2}(\mathscr{G}_{i+1}+\mathscr{G}_{i-1})=0. \tag{4.10}$$

We note that this scheme is not in conservation form. When applied to the duct flow problem (3.12), we obtained solutions with $n=9$, 19 and 39 in which the shock profile was sharp but the shock position was too far to the right.

We believe that the poor results obtained with the above scheme and with scheme (4.5) are due to lack of conservation. In seeking to find a two switching points EDS scheme that conserves, we attempt to construct an artificial viscosity that will lead us to the scheme of Huang [19]. We thus consider the following alternative to (4.4)

$$\bar{A}U_x+\mathscr{G}=\varepsilon(\bar{A}^{-1}(\bar{A}U)_x)_x. \tag{4.11}$$

An exponentially-based approximation scheme for (4.11) is

$$\frac{1}{2h}(\mathscr{F}_{i+1}-\mathscr{F}_{i-1})+\mathscr{G}_i=\frac{\varepsilon}{h^2}\{\bar{A}^{-1}_{i+1/2}\Gamma_{i+1/2}((\bar{A}U)_{i+1}-(\bar{A}U)_i) - \bar{A}^{-1}_{i-1/2}\Gamma_{i-1/2}((A\bar{U})_i-(A\bar{U})_{i-1})\}. \tag{4.12}$$

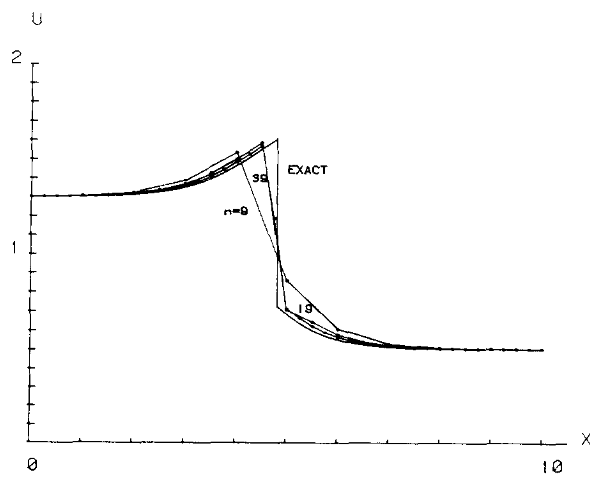

FIG. 4–2. Comparison of solutions to scheme (4.13) with exact solution. Duct flow—system. Two switching point scheme.

Taking the $\varepsilon \to 0$ limit and using the homogeneity property $\bar{A}U = \mathscr{F}$, we obtain the two switching point EDS scheme

$$\frac{1}{2h}(\mathscr{F}_{i+1} - \mathscr{F}_{i-1}) + \mathscr{G}_i = \frac{1}{2h}\{\operatorname{sgn}(\bar{A}_{i+1/2})(\mathscr{F}_{i+1} - \mathscr{F}_i) - \operatorname{sgn}(\bar{A}_{i-1/2})(\mathscr{F}_i - \mathscr{F}_{i-1})\}. \quad (4.13)$$

We see that this scheme is essentially the same as one introduced by Huang [19] and apparently gives shock profiles with at most one mesh point interior to the shock. Huang derived the scheme by analogy with an upwind scheme for Burgers equation. In Fig. 4–2 we show computational results for scheme (4.13) applied to the shock problem (3.12). Comparing with Fig. 4–1 for scheme (4.7), we see that (4.13) does indeed give sharper shock profiles than does (4.7). Moreover, there are no computational oscillations and computational efforts to find multiple solutions similar to those discussed in Section 3 were unsuccessful. (We note, however, that the time-dependent method used for the systems in Section 4 are considered to be less likely than in Newton's method, as used in Section 3, to find multiple solutions [16].)

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